Chapter 2

Basic Tools of the Trade

Exercise 2.1 Find a recurrence relation

(1) (of order 2) for the number of compositions of n with parts in $\mathbb{N}$;

(2) for the number of words of length n on $[k]$;

(3) for the number of permutations of $[n]$;

(4) for the reverse conjugate compositions of n.

Use the iteration method to find explicit formulas for the above recurrence relations.

Solution 2.1 (1) Let a_n be the number of compositions of n with parts in $\mathbb{N}$. Then the number of compositions of n with parts in $\mathbb{N}$ that start with j equals a_{n-j}. Therefore, $a_n = \sum_{j=1}^{n} a_{n-j}$ for $n \geq 1$. Hence, $a_n - a_{n-1} = a_{n-1}$, or equivalently, $a_n = 2a_{n-1}$. Iterating this recurrence and using that $a_0 = 1$, we obtain $a_n = 2^{n-1}$ for all $n \geq 1$.

(2) Let $a_{n,k}$ be the number of words of length n on $[k]$. By considering the first element of a word we obtain that $a_{n,k} = k \cdot a_{n-1,k}$. The initial condition is $a_{0,k} = 1$. Using the iteration method we get that $a_{n,k} = k^n$.

(3) Let a_n be the number of permutations of $[n]$ and let π be any such permutation. By deleting the letter n from π we get a permutation of $[n-1]$. Since there are n possibilities to add back the letter n, we have $a_n = n \cdot a_{n-1}$. The initial condition is $a_0 = 1$, hence $a_n = n(n-1)\cdots 2 \cdot 1 = n!$.

(4) Similar to palindromic compositions, we have to maintain symmetry, so the reverse conjugate compositions of n will be produced from those of $n-2$. For palindromic compositions, we either appended a 1 or increased the last part by 1 (and the same at the beginning of the composition). Since the conjugate is involved, increasing a part in σ results in an extra part of size 1 in the conjugate, which has to be taken into account at the other end. Thus, for every reverse conjugate composition of $n-2$, we create two new reverse conjugate compositions of n as follows: Increase the first part by 1 and append a 1 at the right end; or prepend a 1 at the left end and increase the last part by 1. This process does not create duplicates and is reversible, so if we let r_n be the number of reverse conjugate compositions of n, then we have the following recursion: $r_n = 2r_{n-2}$, with $r_1 = 1$ and $r_2 = 0$. This

implies that there are no reverse conjugate compositions of an even number, and $r_{2m+1} = 2^m r_1 = 2^m$, which is the result proved in Exercise 1.7.

Exercise 2.2 Use Maple[1] or *Mathematica*[2] to find the first 15 terms of the sequences

(1) $a_n = a_{n-1} + 3a_{n-2}$ with $a_0 = 1$ and $a_1 = 2$.

(2) $b_n = \frac{n}{n-1} b_{n-1} + 1$ with $b_1 = 2$.

Solution 2.2 The first 15 terms of the sequence (1) are computed using the Maple code

```
aseq:=proc(n)
  if n<0 then return "seq not defined for negative indices";
  elif n=0 then return 1;
     elif n=1 then return 2;
       else return(aseq(n-1)+3*aseq(n-2));
  end if;
end proc:
seq(aseq(n),n=0..14);
```

or the *Mathematica* code

```
a[0]=1; a[1]=2; a[n_]:=a[n]=a[n-1]+3a[n-2];
Table[a[n],{n,0,14}]
```

The first 15 terms of the sequence (2) are computed using the Maple code

```
aseq:=proc(n)
  if n<1 then return "seq not defined for negative indices";
  elif n=1 then return 2;
    else return(n/(n-1)*aseq(n-1)+1);
  end if;
end proc:
seq(aseq(n),n=1..15);
```

or the *Mathematica* code

```
b[1]=2; b[n_]:=b[n]=n/(n-1)b[n-1]+1;
Table[b[n],{n,1,15}]//N
```

Exercise 2.3 Solve the recurrence relations of Exercise 2.2 using Maple or *Mathematica*.

[1] Maple™ is a registered trademark of Waterloo Maple Software.

[2] *Mathematica® is a registered trademark of Wolfram Research, Inc.*

Solution 2.3 (1) The Maple and *Mathematica* codes are

```
rsolve({a(n)=a(n-1)+3*a(n-2),a(0)=1,a(1)=2},a(n));
```

and

```
RSolve[{a[n]==a[n-1]+3a[n-2],a[0]==1, a[1]==2},a[n],n]
```

(2) The Maple code is

```
rsolve({b(n)=n/(n-1)*b(n-1)+1,b(1)=2},b(n));
```

and the *Mathematica* code is

```
RSolve[{b[n]==n/(n-1)b[n-1]+1,b[1]==2},b[n],n]//Simplify
```

Exercise 2.4 Determine the number of words of length n on the alphabet $[3]$, and derive the generating function for the number of words of length $n-1$ on the alphabet $[3]$.

Solution 2.4 Since there are three choices for each letter, the number of words of length n is given by 3^n. Let a_n denote the number of words of length $n-1$. From the definition of the generating function, we obtain that

$$A(x) = 1 + \sum_{n\geq 1} 3^{n-1}x^n = 1 + x\frac{1}{1-3x} = \frac{1-2x}{1-3x},$$

where the summand 1 accounts for the empty word.

Exercise 2.5 Find recurrence relations for the following counting problems:

(1) The number of words $w = w_1w_2\cdots w_n$ of length n on $\{1,2\}$ satisfying $w_i \geq w_{i+2}$ for all i.

(2) The number of words $w = w_1w_2\cdots w_n$ of length n on $\{1,2,3\}$ satisfying $w_i \geq w_{i+2}$ for all i.

Use the iteration method to find explicit formulas for the above recurrence relations.

Solution 2.5 (1) Let a_n denote the number of words $w = w_1w_2\cdots w_n$ of length n on $\{1,2\}$ satisfying $w_i \geq w_{i+2}$ for all i, and let b_n denote the number of words of length n on $\{1,2\}$ whose entries are in nonincreasing order. Then $a_{2m} = b_m^2$ and $a_{2m+1} = b_mb_{m+1}$. On the other hand, $b_m = 1 + b_{m-1}$ as any such word either starts with a 1 (in which case there is only one such word), or it starts with a 2, followed by any such word of length $m-1$. Iterating the recurrence for b_m and using $b_0 = 1$ gives that $b_m = m+1$, so for $m \geq 0$,

$$a_{2m} = (m+1)^2 \quad \text{and} \quad a_{2m+1} = (m+1)(m+2).$$

(2) Now let a_n denote the number of words $w = w_1w_2\cdots w_n$ of length n on $\{1,2,3\}$ satisfying $w_i \geq w_{i+2}$ for all i, and let b_n denote the number of words of length n on $\{1,2,3\}$ whose entries are in nonincreasing order. We have the same recurrence for a_n as in Part (1), but the recurrence for b_m changes. If the first letter is not a 3, then we are in the situation of Part (1) and there are a total of $m+1$ such nonincreasing words. If the first letter of the word is a 3, then any nonincreasing word can follow, so all together, $b_m = m+1+b_{m-1}$. Iterating this recurrence gives $b_m = (m+1)+m+(m-1)+\ldots+1 = \binom{m+2}{2}$. Thus,

$$a_{2m} = \binom{m+2}{2}^2 \quad \text{and} \quad a_{2m+1} = \binom{m+2}{2}\binom{m+3}{2}$$

for $m \geq 0$.

In fact, it is not difficult to derive a general solution for all k directly, without the recurrence relation. Let $a_{n,k}$ be the number of words w of length n on $[k]$ such that there is no i with $w_i < w_{i+2}$. Thus, w satisfies $w_1 \geq w_3 \geq w_5 \geq \cdots$ and $w_2 \geq w_4 \geq w_6 \geq \cdots$. In the case $n = 2m$ we have $\binom{m+k-1}{k-1}$ possibilities to choose the letters at the odd and even locations within the word, respectively (since $\binom{m+k-1}{k-1}$ gives the number of solutions of $x_1+x_2+\cdots+x_k = m$ where $x_i \geq 0$ denotes the number of parts i in the selected positions). Once the letters have been selected, there is only one way to place them in nonincreasing order, thus $a_{2m,k} = \binom{m+k-1}{k-1}^2$. Similarly, in the case $n = 2m+1$ we get that $a_{2m+1,k} = \binom{m+1+k-1}{k-1}\binom{m+k-1}{k-1}$. Hence, for all $n \geq 0$ and $k \geq 1$,

$$a_{n,k} = \binom{[(n+1)/2]+k-1}{k-1}\binom{[n/2]+k-1}{k-1},$$

which gives the results of Parts (1) and (2) as a special case.

Exercise 2.6

(1) Find an explicit formula for the number of Carlitz compositions of n in $\{1,2\}$.

(2) A word is called *Carlitz* if it does not contain two consecutive letters that are the same. For example, 121 and 212 are the only Carlitz words on $\{1,2\}$ of length three. Find an explicit formula for the number of Carlitz words on $\{1,2\}$ of length n.

Solution 2.6 (1) Let a_n (respectively b_n) be the number of Carlitz compositions of n with parts in $\{1,2\}$ that start with 1 (respectively 2). Thus $a_n = b_{n-1}$ and $b_n = a_{n-2}$ for all $n \geq 3$, and therefore $a_n = a_{n-3}$ and $b_n = b_{n-3}$ for $n > 3$. If c_n is the number of Carlitz compositions of n with parts in $\{1,2\}$, then $c_n = a_n + b_n = a_{n-3} + b_{n-3} = c_{n-3}$. The initial conditions are $c_0 = c_1 = c_2 = 1$ and $c_3 = 2$. Hence, $c_n = 2$ for $n \equiv 0 \pmod 3$ and 1 otherwise. Note that there is an easy combinatorial proof for the explicit formula. When n is a multiple of three, then the composition can start with either a 1 or a 2, and ends with the opposite value. If $n \equiv 1 \pmod 3$, then there is one more 1 than there are 2s, and the composition has to start with a 1. For $n \equiv 2 \pmod 3$, there is one more 2 than there are 1s.

(2) Let a_n (respectively b_n) be the number of Carlitz words of length n on $\{1, 2\}$ that start with 1 (respectively 2). If a word starts with 1, the next letter has to be a 2 and vice versa. Therefore $a_n = b_{n-1}$ and $b_n = a_{n-1}$ for all $n \geq 2$. If t_n is the number of Carlitz words of length n on $\{1, 2\}$, then $t_n = a_n + b_n = a_{n-1} + b_{n-1} = t_{n-1}$. The initial conditions are $t_0 = 1$ and $t_1 = 2$. Hence $t_n = 2$ for all $n \geq 1$. The combinatorial explanation is similar to Part (1). Since the word is Carlitz, the letters 1 and 2 have to alternate, and the word can either start with 1 or with 2, thus $t_n = 2$ for $n \geq 1$.

Exercise 2.7 Prove the identity $\sum_{i=1}^{n+1} F_i = F_{n+3} - 1$ for $n \geq 0$ (where F_n is the n-th Fibonacci number) by using rules for generating functions.

Solution 2.7 By (2.8), the generating function for the Fibonacci sequence is given by $F(x) = x/(1-x-x^2)$. To show the equality, we derive the generating functions for each side of the identity to be proven and check that they are equal. On the left hand side, we have a partial sum of the Fibonacci sequence shifted by 1, so its generating function is given by (see Rules 2.45 and 2.52)

$$\frac{1}{1-x}\frac{F(x)}{x},$$

while the generating function for the right-hand side of the identity is given by

$$\frac{F(x) - F_1 x - F_2 x^2}{x^3} - \frac{1}{1-x}.$$

Substituting $F(x)$ shows equality of the generating functions.

Exercise 2.8 Derive the explicit formula for the Lucas sequence.

Solution 2.8 The Lucas sequence satisfies $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_0 = 2$ and $L_1 = 1$. Using Example 2.25, we get that

$$L_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

with initial conditions

$$c_1 + c_2 = 2 \quad \text{and} \quad c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1,$$

which yields $c_1 = c_2 = 1$. Thus,

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n = \alpha^n + \beta^n.$$

Exercise 2.9 Prove by induction that the explicit formulas for the Fibonacci and Lucas sequences produce integer values.

Solution 2.9 Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then $\alpha - \beta = \sqrt{5}$, $\alpha\beta = -1$, $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, and $L_n = \alpha^n + \beta^n$ (see Exercise 2.8). For $n = 0$ we obtain the integer values $F_0 = 0$ and $L_0 = 2$. We will prove simultaneously by induction on n that both F_n and L_n are integers. Assuming the hypothesis to be true for n, we obtain

$$\begin{aligned} F_{n+1} &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{(\alpha - \beta)(\alpha^n + \beta^n) + \alpha\beta(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta} \\ &= L_n - F_{n-1} \end{aligned}$$

and

$$\begin{aligned} L_{n+1} &= \alpha^{n+1} + \beta^{n+1} = (\alpha - \beta)(\alpha^n - \beta^n) + \alpha\beta(\alpha^{n-1} + \beta^{n-1}) \\ &= 5F_n - L_{n-1}, \end{aligned}$$

and therefore, the formulas always produce integer values.

Exercise 2.10 Prove Theorem 2.22.

Solution 2.10 Let $\alpha_0 = 1$. Since ξ is a root of $\Delta(x)$ with multiplicity m, then ξ is also a root of $\Delta^{(s)}(x)$, the s-th derivative of $\Delta(x)$ for all $s = 0, 1, 2, \ldots, m-1$. Substituting $a_n = n^i\xi^n$ into (2.3), expanding (binomial theorem) and collecting terms according to the summation index we get that

$$\sum_{j=0}^{r} \alpha_j (n - r + r - j)^i \xi^{n-j} = \sum_{k=0}^{i} \xi^{n-r} \binom{i}{k} (n - r)^{i-k} \sum_{j=0}^{r} (r - j)^k \alpha_j \xi^{r-j}.$$

Let $B_k(x) := \sum_{j=0}^{r} (r - j)^k \alpha_j x^{r-j}$. It is easy to see that $B_0(x) = \Delta(x)$ and $B_k(x) = x \cdot B'_{k-1}(x)$ for $k \geq 1$. Since ξ is a root of $\Delta(x)$ with multiplicity m, $B_0(\xi) = 0$, and by induction we get that $B_1(\xi) = B_2(\xi) \ldots = B_{m-1}(\xi) = 0$. Thus, $\sum_{j=0}^{r} \alpha_j (n - r + r - j)^i \xi^{n-j} = 0$, that is, $n^i\xi^n$ satisfies (2.3) for all $i = 0, 1, \ldots, m - 1$.

Exercise 2.11 Prove Rules 2.45 and 2.56 either by induction on k or by using the definition of the generating function.

Solution 2.11 For the ordinary generating function we have

$$\sum_{n\geq 0} a_{n+k} x^n = \frac{1}{x^k} \sum_{n\geq 0} a_{n+k} x^{n+k} = \frac{1}{x^k} \sum_{n\geq k} a_n x^n = \frac{A(x) - \sum_{j=0}^{k-1} a_j x^j}{x^k}.$$

For the exponential generating function, we have

$$(D^kE)(x) = \sum_{n=k}^{\infty} a_n n(n-1)\cdots(n-k+1)\frac{x^{n-k}}{n!} = \sum_{n=k}^{\infty} a_n \frac{x^{n-k}}{(n-k)!}$$
$$= \sum_{n=0}^{\infty} a_{n+k}\frac{x^n}{n!}.$$

Exercise 2.12 Compute the generating function for the sequence $\{a_n\}_{n\geq 0}$, where $a_0 = 1$ and $a_n = F_{2n}$ for $n \geq 1$.

Solution 2.12 The generating function $F(x)$ of the sequence $\{F_n\}_{n\geq 0}$ is given by $\frac{x}{1-x-x^2}$ (see Example 2.29). Thus, the generating function for the sequence $\{a_n\}_{n\geq 0}$ equals

$$1 + \sum_{n\geq 0} F_{2n}x^n = 1 + \sum_{n\geq 0} F_{2n}(\sqrt{x})^{2n} = 1 + \frac{1}{2}(F(\sqrt{x}) + F(-\sqrt{x}))$$
$$= 1 + \frac{x}{(1-\sqrt{x}-x)(1+\sqrt{x}-x)} = 1 + \frac{x}{1-3x+x^2}.$$

Exercise 2.13 Let $C(m;n)$ denote the number of compositions of n with m parts in $\mathbb{N}$. Prove that for fixed $m \geq 1$, $\sum_{n\geq 0} C(m;n)x^n = \dfrac{x^m}{(1-x)^m}$.

Solution 2.13 The generating function for a single nonzero part is given by $\sum_{n\geq 1} x^n = x/(1-x)$. Therefore, a composition $\sigma \in \mathcal{C}_{m,n}$ (as sum of m parts) has a generating function that is the m-fold convolution of the generating functions of the single part by Rule 2.51.

Exercise 2.14

(1) Derive the generating function for the number of compositions of n in $\{1,2\}$ that avoid the substring 11. The recurrence relation for this sequence was derived in Example 2.17.

(2) The first few values of the sequence are given by 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16 and 21. Checking this sequence in the Online Encyclopedia of Integer Sequences [21] produces a match with the Padovan sequence A000931. However, the sequence for the number of compositions is shifted in relation to the Padovan sequence. Use the generating function given for the Padovan sequence and apply Rule 2.45 to derive the generating function for the number of compositions in $\{1,2\}$ that avoid the substring 11. Compare your answer to Part (1).

Solution 2.14 (1) Let a_n be the number of compositions with parts in $\{1,2\}$ that avoid the substring 11. Multiplying the recurrence relation of Example 2.17 by x^n and summing over $n \geq 3$, we obtain that

$$A(x) - x^2 - x - 1 = x^2(A(x) - 1) + x^3 A(x),$$

which is equivalent to $A(x) = \frac{1+x}{1-x^2-x^3}$.

(2) Since the offset is 0, the given sequence starts with P_0, and therefore, $(1-x^2)/(1-x^2-x^3) \overset{\text{ops}}{\leftrightarrow} \{P_n\}_{n\geq 0}$ with initial conditions $P_0 = P_3 = 1$, $P_1 = P_2 = P_4 = 0$. Thus using Rule 2.45, the generating function for $\{a_n\}_{n\geq 0}$ is given by

$$\frac{\frac{(1-x^2)}{(1-x^2-x^3)} - 1 - x^3}{x^5} = \frac{1+x}{1-x^2-x^3},$$

which is identical to the result in Part (1). Moral of the story: if the sequence of interest is shifted from a sequence with known generating function then there are two ways to compute the desired generating function.

Exercise 2.15 Find the recurrence relation and the generating function for the number of compositions of n in $\{1,2\}$ without d consecutive 1s. (Hint: condition on the position of the first 2.)

Solution 2.15 Let $\mathcal{A}_{d,n}$ be the set of all compositions of n with parts in $\{1,2\}$ without d consecutive 1s. Define $a(n,j)$ to be the number of compositions in $\mathcal{A}_{d,n}$ that start with exactly j ones (more specifically, j 1s followed by a 2), and let a_n denote the number of compositions without d consecutive 1s. Thus, $a(n,j) = a_{n-j-2}$, where a_n is the number of compositions of n in $\{1,2\}$ without d consecutive 1s. Counting the compositions in $\mathcal{A}_{d,n}$ according to how they start we have that

$$\begin{aligned} a_n &= a(n,0) + a(n,1) + a(n,2) + \ldots + a(n,d-1) \\ &= a_{n-2} + a_{n-3} + \ldots + a_{n-d-1}, \end{aligned}$$

for $n \geq d+1$. For $n < d$, a_n equals the number of compositions with 1s and 2s, that is, $a_n = F_{n+1}$ (see Example 2.9). For $n = d$, $a_d = F_{d+1} - 1$ as we have to exclude the composition consisting of all 1s. Setting up the recurrence for the generating function $A(x)$ gives

$$\begin{aligned} &A(x) - \sum_{j=0}^{d} F_{j+1}x^j + x^d = \\ &\quad x^2\left(A(x) - \sum_{j=0}^{d-2} F_{j+1}x^j\right) + x^3\left(A(x) - \sum_{j=0}^{d-3} F_{j+1}x^j\right) + \cdots + x^{d+1}A(x), \end{aligned}$$

which gives

$$(1 - x^2 - x^3 - \cdots - x^{d+1})A(x) = -x^d + \sum_{j=0}^{d} F_{j+1}x^j - \sum_{i=2}^{d} x^i \sum_{j=0}^{d-i} F_{j+1}x^j.$$

Rearranging the double sum and using Exercise 2.7 gives

$$\sum_{i=2}^{d} x^i \sum_{j=0}^{d-i} F_{j+1} x^j = \sum_{j=2}^{d} x^j \left(\sum_{i=1}^{j-1} F_j \right) = \sum_{j=2}^{d} x^j \left(F_{j+1} - 1 \right).$$

Therefore

$$A(x) = \frac{1 + x + \cdots + x^{d-1}}{1 - x^2 - x^3 - \cdots - x^{d+1}},$$

which reduces to the generating function for Exercise 2.14 for $d = 2$.

Exercise 2.16 Derive a recurrence relation for the Bell numbers B_n (see Definition 2.59).

Solution 2.16 We count the number of partitions of a set of $n + 1$ elements according to the size of the set containing the $(n+1)$-st element. If the set has size j for $1 \le j \le n + 1$, then there are $\binom{n}{j-1}$ choices for the n other elements of that set. The remaining $n + 1 - j$ elements can be partitioned in B_{n+1-j} ways. Thus,

$$\begin{aligned} B_{n+1} &= \sum_{j=1}^{n+1} \binom{n}{j-1} B_{n+1-j} = \sum_{j=1}^{n+1} \binom{n}{n+1-j} B_{n+1-j} \\ &= \sum_{k=0}^{n} \binom{n}{k-j} B_k. \end{aligned}$$

Exercise 2.17 A *smooth* word is a word in which the difference between any two adjacent letters is either -1, 0 or 1. Find an explicit formula for the generating function for the number of smooth words of length n on the alphabet $[3]$. Use either Maple or *Mathematica* to find an explicit formula for the number of smooth words of length n on the alphabet $[3]$.

Solution 2.17 Let $f(x)$ be the generating function for the number of smooth words of length n on the alphabet $[3]$. Then

$$f(x) = 1 + f(1|x) + f(2|x) + f(3|x),$$

where $f(i|x)$ is the generating function for the number of smooth words of length n on the alphabet $[3]$ such that the first letter is i. It is not hard to see that $f(2|x) = xf(x)$, $f(1|x) = x + xf(2|x) + xf(1|x)$, and $f(3|x) = x + xf(2|x) + xf(3|x)$, which implies that

$$f(1|x) = f(3|x) = \frac{x}{1-x} + \frac{x^2}{1-x} f(x).$$

Thus $f(x)$ satisfies

$$f(x) = 1 + xf(x) + 2\frac{x}{1-x} + 2\frac{x^2}{1-x} f(x),$$

which implies that $f(x) = \dfrac{1+x}{1-2x-x^2}$. Therefore, the number of smooth words of length n on the alphabet [3] is given by

$$\frac{1}{2}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right).$$

This explicit formula can be found by using either the Maple code

```
with(genfunc):
rgf_expand((1+x)/(1-2*x-x^2), x, n);
```

or the *Mathematica* code

```
SeriesCoefficient[(1+x)/(1-2x-x^2),{x,0,n}]
```

Exercise 2.18 A *strictly smooth* word is a word in which the difference between any two adjacent letters is either -1 or 1. Find an explicit formula for the generating function for the number of strictly smooth words of length n on the alphabet [3]. Use either *Mathematica* or Maple to find an explicit formula for the number of strictly smooth words of length n on the alphabet [3].

Solution 2.18 Let $f(i|x)$ be the generating function for the number of strictly smooth words of length n on the alphabet [3] such that the first letter is i. Using arguments similar to those in the solution of Exercise 2.17 we get that the generating function $f(x)$ for the number of strictly smooth words of length n on [3] satisfies

$$f(x) = 1 + f(1|x) + f(2|x) + f(3|x)$$

where $f(2|x) = x + xf(1|x) + xf(3|x)$, $f(1|x) = x + xf(2|x)$ and $f(3|x) = x + xf(2|x)$. This implies that $f(2|x) = \dfrac{x+2x^2}{1-2x^2}$ and therefore

$$f(x) = 1 + 2x + (1+2x)f(2|x) = \frac{(1+x)(1+2x)}{1-2x^2},$$

from which we obtain that the number of strictly smooth words of length n on [3] is given by

$$\frac{4+3\sqrt{2}}{4}\sqrt{2}^n + \frac{4-3\sqrt{2}}{4}(-\sqrt{2})^n.$$

This explicit formula can be obtained by using the Maple code

```
with(genfunc):
rgf_expand((1+x)*(1+2*x)/(1-2*x^2), x, n);
```

or the *Mathematica* code

```
SeriesCoefficient[(1+x)(1+2x)/(1-2x^2),{x,0,n}]//Simplify
```

Note that sometimes the answers given by Maple or *Mathematica* have to be "massaged" a bit to result in a nice formula.

Exercise 2.19 Find an explicit formula for the generating function for the number of smooth compositions (see Exercise 1.12) of n with parts in [3].

Solution 2.19 Let $f(x)$ be the generating function for the number of smooth compositions of n with parts in [3]. Then

$$f(x) = 1 + f(1|x) + f(2|x) + f(3|x),$$

where $f(i|x)$ is the generating function for the number of smooth compositions of n with parts in [3] such that the first part is i. It is not hard to see that $f(2|x) = x^2f(x)$, $f(1|x) = x+xf(2|x)+xf(1|x)$, and $f(3|x) = x^3+x^3f(2|x)+x^3f(3|x)$, which implies that

$$f(1|x) = \frac{x}{1-x} + \frac{x^3}{1-x}f(x) \quad \text{and} \quad f(3|x) = \frac{x^3}{1-x^3} + \frac{x^5}{1-x^3}f(x).$$

Thus

$$f(x) = \frac{(1+x)(1+x^2)}{1-x^2-2x^3-x^4-x^5}.$$

Exercise 2.20 Find an explicit formula for the generating function for the number of strictly smooth compositions (see Exercise 1.12) of n with parts in [3].

Solution 2.20 As in the solution to Exercise 2.19 we obtain that the generating function $f(x)$ for the number of strictly smooth compositions of n with parts in [3] is given by $f(x) = 1 + f(1|x) + f(2|x) + f(3|x)$, but now $f(2|x) = x^2 + x^2f(1|x) + x^2f(3|x)$, $f(1|x) = x + xf(2|x)$, and $f(3|x) = x^3 + x^3f(2|x)$. This implies that $f(2|x) = \frac{x^2+x^3+x^5}{1-x^3-x^5}$. Hence

$$f(x) = 1 + x + x^3 + (1+x+x^3)f(2|x) = \frac{(1+x^2)(1+x+x^3)}{1-x^3-x^5}.$$

Exercise 2.21 A *k-ary tree* is a plane tree in which each vertex has either out-degree 0 or k (see Definitions 7.2 and 7.20). A vertex is said to be *internal* if its out-degree is k. Use the LIF to find an explicit formula for the number of k-ary trees on n internal vertices.

Solution 2.21 Let $h(x)$ be the generating function for the number of k-ary trees with n internal vertices. The k-ary tree is either empty or there is at least one internal vertex with k subtrees that are also k-ary trees. Thus $h(x) = 1 + xh^k(x)$. To apply the LIF, let $u(x) = h(x) - 1$, $\phi(u) = (u+1)^k$, and $f(u) = u$. Then the assumptions of Theorem 2.63 are satisfied, and the Lagrange Inversion formula gives

$$[x^n]u(x) = \frac{1}{n}[u^{n-1}](u+1)^{kn} = \frac{1}{n}\binom{kn}{n-1} = \frac{1}{1+(k-1)n}\binom{kn}{n}.$$

Thus, the number of k-ary trees with n internal vertices is $\frac{1}{1+(k-1)n}\binom{kn}{n}$ for $n \geq 1$ (since h and u differ only by the constant term), and the formula also holds for $n = 0$.

Exercise 2.22 Go to the On-Line Encyclopedia of Integer Sequences [21] and look up the functional equation for the large Schröder numbers (sequence A006318). Verify that it has the form of (2.11) and derive its continued fraction form.

Solution 2.22 The generating function for the large Schröder numbers satisfies $(1-x)S(x) - xS(x)^2 = 1$, which is equivalent to $S(x) = 1 + xS(x) + xS(x)^2$ and has the form of (2.11). Therefore, the generating function for the large Schröder numbers can be written as

$$S(x) = \cfrac{1}{1 - x - \cfrac{x}{1 - x - \cfrac{x}{1 - x - \ddots}}}.$$

Exercise 2.23* Let $f_{m,l}(x,y) = \left(\frac{(1-x-y)^m}{(1-2x-(1-x)y)^{m+1}}\right)$. Show that

$$[y^\ell]f_{m,l}(x,y) = \frac{(1-x)^{\ell-m}}{(1-2x)^{\ell+1}} \sum_{i\geq 0} \binom{\ell+i}{i}\binom{m}{i} \frac{x^{2i}}{(1-2x)^i}.$$

Hint: $[y^\ell]f_{m,l}(x,y) = \frac{(1-x)^\ell}{(1-2x)^\ell}[y^\ell]f_{m,l}\big(x, \frac{(1-2x)}{(1-x)}y\big)$.

Solution 2.23 Let $f_{m,l}(x,y) = \left(\frac{(1-x-y)^m}{(1-2x-(1-x)y)^{m+1}}\right)$. Then

$$
\begin{aligned}
[y^\ell]f_{m,l}(x,y) &= \frac{(1-x)^\ell}{(1-2x)^\ell}[y^\ell]\left[\frac{\left(1-x-\frac{(1-2x)}{(1-x)}y\right)^m}{\left(1-2x-(1-x)\frac{(1-2x)}{(1-x)}y\right)^{m+1}}\right] \\
&= \frac{(1-x)^{\ell-m}}{(1-2x)^{\ell+m+1}}[y^\ell]\left[\frac{((1-x)^2-(1-2x)y)^m}{(1-y)^{m+1}}\right] \\
&= \frac{(1-x)^{\ell-m}}{(1-2x)^{\ell+m+1}}[y^\ell]\left[\frac{((1-2x)(1-y)+x^2)^m}{(1-y)^{m+1}}\right] \\
&= \frac{(1-x)^{\ell-m}}{(1-2x)^{\ell+m+1}}[y^\ell]\left[\frac{\left((1-2x)+\frac{x^2}{(1-y)}\right)^m}{(1-y)}\right] \\
&= \frac{(1-x)^{\ell-m}}{(1-2x)^{\ell+m+1}}[y^\ell]\left[\sum_{j=0}^{m}\binom{m}{j}\frac{x^{2j}(1-2x)^{m-j}}{(1-y)^{j+1}}\right] \\
&= \frac{(1-x)^{\ell-m}}{(1-2x)^{\ell+1}}[y^\ell]\left[\sum_{j=0}^{m}\binom{m}{j}\frac{x^{2j}}{(1-2x)^j}\sum_{i\geq 0}\binom{i+j}{j}y^i\right] \\
&= \frac{(1-x)^{\ell-m}}{(1-2x)^{\ell+1}}\sum_{j=0}^{m}\binom{m}{j}\frac{x^{2j}}{(1-2x)^j}\binom{\ell+j}{j}.
\end{aligned}
$$

Chapter 3

Compositions

Exercise 3.1 Let $A(x,y,\vec{q}) = \sum_{n\geq 0}\sum_{\sigma\in\mathcal{C}_{n,m}} x^n y^m \prod_{i=1}^{\infty} q_i^{r_i(\sigma)}$ be the generating function for the number of compositions of n with r_i parts of size i and $\sum_i r_i = m$. Prove that

$$A(x,y,\vec{q}) = \frac{1}{1 - y\sum_{i=1}^{\infty} q_i x^i}.$$

Solution 3.1 See the proof of Theorem 2.1 in [14].

Exercise 3.2 Let $A(x,y,\vec{q}) = \sum_{n\geq 0}\sum_{\sigma\in\mathcal{P}_{n,m}} x^n y^m \prod_{i=1}^{\infty} q_i^{r_i(\sigma)}$ be the generating function for the number of palindromic compositions of n with r_i parts of size i and $\sum_i r_i = m$. Prove that

$$A(x,y,\vec{q}) = \frac{1 + y\sum_{i=1}^{\infty} q_i x^i}{1 - y^2\sum_{i=1}^{\infty} q_i^2 x^{2i}}.$$

Solution 3.2 See the proof of Theorem 2.6 in [14].

Exercise 3.3* Find an explicit formula for $\dfrac{C_{\mathbb{N}\setminus\{1\}}(n)}{C_{\mathbb{N}}(n)}$ and determine its limit when $n \to \infty$.

Solution 3.3 The number of compositions of n with m parts in $\mathbb{N}\setminus\{1\}$ is the same as the number of compositions of $n-m$ with m parts in $\mathbb{N}$ (by adding 1 to each part). Thus

$$\frac{C_{\mathbb{N}\setminus\{1\}}(n)}{C_{\mathbb{N}}(n)} = \frac{\sum_{m=1}^{n} C_{\mathbb{N}}(m; n-m)}{C_{\mathbb{N}}(n)}.$$

Using Theorem 3.3 we get that

$$\frac{C_{\mathbb{N}\setminus\{1\}}(n)}{C_{\mathbb{N}}(n)} = \frac{\sum_{m=1}^{n}\binom{n-m-1}{m-1}}{2^{n-1}} = \frac{F_{n-1}}{2^{n-1}},$$

where F_n is the n-th Fibonacci number. Since $F_{n-1} = \frac{1}{\sqrt{5}}(\alpha^{n-1} - \beta^{n-1})$ with $1 < \alpha < 2$ and $|\beta| < 1$, the ratio tends to zero as $n \to \infty$.

Exercise 3.4 Find the generating function for the number of compositions of n with m parts in $\mathbb{N}$ in which no part is unique (or equivalently, every part appears at least twice). (Hint: use the exponential generating function for y, that is, $f(x,y) = \sum_{m\geq 0} f(m;x)\frac{y^m}{m!}$, where $f(m;x)$ is the ordinary generating function for the number of compositions with m parts in which no part is unique.)

Solution 3.4 Let $f_k(m;x)$ be the generating function for the number of compositions of n with m parts in $[k]$ in which no part is unique. Then, enumerating according to the number of parts k, we obtain

$$f_k(m;x) = \sum_{j\neq 1} \binom{m}{j} f_{k-1}(m-j;x)x^{kj}.$$

Now let $f_k(x,y) = \sum_{m\geq 0} f_k(m;x)y^m/m!$, then

$$\begin{aligned} f_k(x,y) = &\sum_{m\geq 0}\sum_{j=0}^{m} \binom{m}{j} f_{k-1}(m-j;x)x^{kj}\frac{y^m}{m!} \\ &- \sum_{m\geq 0} \binom{m}{1} f_{k-1}(m-1;x)x^{k}\frac{y^m}{m!}. \end{aligned}$$

Changing the order of summation for the first sum and re-indexing both sums gives that

$$f_k(x,y) = (e^{x^k y} - x^k y)f_{k-1}(x,y).$$

Using induction on k with $f_0(x,y) = 1$, we obtain that

$$f_k(x,y) = \prod_{j=1}^{k} (e^{x^j y} - x^j y).$$

The number of compositions of n with m parts in $\mathbb{N}$ in which no part is unique can then be obtained as the coefficient of $x^n y^m/m!$ in the generating function $f_k(x,y)$.

Exercise 3.5 A composition σ is said to be *odd-Carlitz* (*respectively, even-Carlitz*) if it is Carlitz and all its parts are odd (respectively, even) numbers. Prove that

(1) the generating function for the number of odd-Carlitz compositions of n with m parts in $\mathbb{N}$ is given by

$$\frac{1}{1 - \sum_{i\geq 0} \frac{x^{2i+1}y}{1+x^{2i+1}y}};$$

(2) the generating function for the number of even-Carlitz compositions of n with m parts in $\mathbb{N}$ is given by

$$\frac{1}{1-\sum_{i\geq 0}\frac{x^{2i+2}y}{1+x^{2i+2}y}}.$$

Solution 3.5 Let

$$f(x,y)=\sum_{\sigma}x^{\mathrm{ord}(\sigma)}y^{\mathrm{par}(\sigma)}\quad\text{and}\quad g(x,y)=\sum_{\sigma}x^{\mathrm{ord}(\sigma)}y^{\mathrm{par}(\sigma)},$$

where the sum is over all odd-Carlitz and even-Carlitz compositions, respectively. Let $\sigma=\sigma_1\cdots\sigma_m$ be any composition with m parts. Note that σ is an even-Carlitz composition of n if and only if $(\sigma_1-1)\cdots(\sigma_m-1)$ is an odd-Carlitz composition of $n-m$, thus $g(x,y)=f(x,xy)$.

Let $f(i|x,y)$ be the generating function for the number of odd-Carlitz compositions σ of n with m parts in $\mathbb{N}$ that start with i. Clearly, $f(x,y)=1+\sum_{i\geq 0}f(2i+1|x,y)$. It is not hard to see that

$$\begin{aligned}f(2i+1|x,y)&=x^{2i+1}y\sum_{j\neq i}f(2j+1|x,y)\\&=x^{2i+1}yf(x,y)-x^{2i+1}yf(2i+1|x,y),\end{aligned}$$

or equivalently, $f(2i+1|x,y)=\frac{x^{2i+1}y}{1+x^{2i+1}y}f(x,y)$. Hence, the generating function $f(x,y)$ is given by

$$f(x,y)=\frac{1}{1-\sum_{i\geq 0}\frac{x^{2i+1}y}{1+x^{2i+1}y}},$$

which implies that

$$g(x,y)=\frac{1}{1-\sum_{i\geq 0}\frac{x^{2i+2}y}{1+x^{2i+2}y}}.$$

Exercise 3.6 Prove that the generating function for the number of palindromic Carlitz compositions of n with m parts is given by

$$CP(x,y)=\sum_{m\geq 0}CP(m;x)y^m=1+\frac{\sum_{i\geq 1}\frac{x^iy}{1+x^{2i}y^2}}{1-\sum_{i\geq 1}\frac{x^{2i}y^2}{1+x^{2i}y^2}}.$$

Solution 3.6 Let $C\!P(i|x,y)$ be the generating function for the number of palindromic Carlitz compositions of n with m parts in $\mathbb{N}$ that start with i. Clearly, $C\!P(x,y) = 1 + \sum_{i\geq 1} C\!P(i|x,y)$. Then

$$\begin{aligned} C\!P(i|x,y) &= x^i y + x^{2i}y^2 \sum_{j\neq i} C\!P(j|x,y) \\ &= x^i y + x^{2i}y^2 C\!P(x,y) - x^{2i}y^2 C\!P(i|x,y) - x^{2i}y^2, \end{aligned}$$

or equivalently,

$$C\!P(i|x,y) = \frac{x^i y}{1+x^{2i}y^2} + \frac{x^{2i}y^2}{1+x^{2i}y^2} C\!P(x,y) - \frac{x^{2i}y^2}{1+x^{2i}y^2}.$$

Thus by $C\!P(x,y) = 1 + \sum_{i\geq 1} C\!P(i|x,y)$ we obtain that

$$\left(1 - \sum_{i\geq 1} \frac{x^{2i}y^2}{1+x^{2i}y^2}\right)(C\!P(x,y) - 1) = \sum_{i\geq 1} \frac{x^i y}{1+x^{2i}y^2}.$$

Hence the generating function $C\!P(x,y)$ is given by

$$C\!P(x,y) = 1 + \frac{\sum_{i\geq 1} \frac{x^i y}{1+x^{2i}y^2}}{1 - \sum_{i\geq 1} \frac{x^{2i}y^2}{1+x^{2i}y^2}}.$$

You may check that letting $y = 1$ gives the result of Theorem 3.9, namely

$$C\!P(x,1) = 1 + \frac{\sum_{i\geq 1} \frac{x^i}{1+x^{2i}}}{1 - \sum_{i\geq 1} \frac{x^{2i}}{1+x^{2i}}}.$$

Exercise 3.7

(1) Find an explicit formula for the number of times the summand k occurs in all palindromic compositions of n with parts in $\mathbb{N}$.

(2) More generally, for any ordered subset A of $\mathbb{N}$, find the generating function for the number of times that the coefficient a_i occurs in all palindromic compositions with parts in A.

Solution 3.7 (1) See [5, Theorem 6].

(2) See [6, Theorem 1.4]

Exercise 3.8 Find an explicit formula for the number of Carlitz words (words in which no two adjacent letters are the same) of length n on the alphabet $[k]$.

Solution 3.8 Once we choose the first letter in the Carlitz word, we have $k-1$ possibilities for each letter that follows (namely any letter that is not equal to the letter that preceded it). Altogether, there are $k(k-1)^{n-1}$ Carlitz words.

Exercise 3.9 Derive the generating function for the number of rises in all compositions with parts in $\{1,2\}$ from the recurrence relation given in Theorem 3.11.

Solution 3.9 Multiplying the recurrence relation

$$r(n+1) = r(n) + r(n-1) + F_{n-1}$$

by x^{n+1}, summing over all $n \geq 1$ and re-indexing the sums as necessary we get that

$$\sum_{n\geq 2} r(n)x^n = x\sum_{n\geq 1} r(n)x^n + x^2\sum_{n\geq 0} r(n)x^n + x^2\sum_{n\geq 0} F_n x^n.$$

Using the initial conditions $r(1) = r(0) = 0$ and the generating function for the Fibonacci sequence from Example 2.29 yields

$$(1-x-x^2)\sum_{n\geq 0} r(n)x^n = x^2\sum_{n\geq 0} F_n x^n = \frac{x^3}{1-x-x^2},$$

which implies that the generating function for the sequence $\{r(n)\}_{n\geq 0}$ is given by

$$\sum_{n\geq 0} r(n)x^n = \frac{x^3}{(1-x-x^2)^2}.$$

Exercise 3.10 Fill in the details of the proof of Part (2) of Theorem 3.12.

Solution 3.10 The general solution is of the form $h(n)+p(n)$ where $h(n) = c_1\alpha^n + c_2\beta^n$ and $p(n) = c_3 n\alpha^n + c_4 n\beta^n$ with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. We first substitute $p(n)$ and $F_{n-1} = \frac{1}{\sqrt{5}}(\alpha^{n-1} - \beta^{n-1})$ into the recurrence relation

$$a(n+1,1) = a(n,1) + a(n-1,1) + F_{n-1}$$

to obtain that

$$\begin{aligned} &c_3(n+1)\alpha^{n+1} + c_4(n+1)\beta^{n+1} \\ &\quad = c_3 n\alpha^n + c_4 n\beta^n + c_3(n-1)\alpha^{n-1} + c_4(n-1)\beta^{n-1} + \frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{5}}, \end{aligned}$$

or equivalently,

$$\begin{aligned} &c_3 n\alpha^{n-1}(\alpha^2-\alpha-1) + c_4 n\beta^{n-1}(\beta^2-\beta-1) \\ &\quad + c_3\alpha^{n-1}(\alpha^2+1) + c_4\beta^{n-1}(\beta^2+1) = \frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{5}}. \end{aligned}$$

Because $\alpha^2 - \alpha - 1 = \beta^2 - \beta - 1 = 0$ we obtain that

$$c_3\alpha^{n-1}(\alpha^2+1) + c_4\beta^{n-1}(\beta^2+1) = \frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{5}},$$

which is equivalent to

$$\alpha^{n-1}(c_3\alpha^2 - \frac{1}{\sqrt{5}} + c_3) + \beta^{n-1}(c_4\beta^2 + \frac{1}{\sqrt{5}} + c_4) = 0$$

for all $n \geq 1$. Therefore,

$$c_3\alpha^2 - \frac{1}{\sqrt{5}} + c_3 = c_4\beta^2 + \frac{1}{\sqrt{5}} + c_4 = 0.$$

Solving for c_3 and c_4 gives $c_3 = \frac{\sqrt{5}-1}{10}$ and $c_4 = -\frac{1+\sqrt{5}}{10}$. So we have

$$a(n,1) = c_1\alpha^n + c_2\beta^n + \frac{\sqrt{5}-1}{10}n\alpha^n - \frac{1+\sqrt{5}}{10}n\beta^n.$$

Applying the initial conditions $a(2,1) = 2$ and $a(3,1) = 3$ results in a system of two equations in the two unknowns c_1 and c_2 which can be solved using the *Mathematica* code

```
a = (1 + Sqrt[5])/2; b = (1 - Sqrt[5])/2;
Solve[{2==c1*a^2+c2*b^2+(-1+Sqrt[5])/10*2a^2+(-1-Sqrt[5])/10*2b^2,
       3==c1*a^3+c2*b^3+(-1+Sqrt[5])/10*3a^3+(-1-Sqrt[5])/10*3b^3},
      {c1,c2}]//Simplify
```

which results in the output

```
{{c1->1-7/(5 Sqrt[5]),c2->1+7/(5 Sqrt[5])}}
```

as was to be shown.

Exercise 3.11 Write a program that uses the recursive creation described in Theorem 3.12 to create the compositions of n with odd parts.

Solution 3.11 Here is a *Mathematica* program to do this. Individual compositions are represented by the ordered list of their parts. We start with the initialization for the odd compositions of 1 and 2, namely the compositions 1 and 11.

```
OddComps[1]={{1}}; OddComps[2]={{1, 1}};
```

We define two functions, **App1** and **Inc**, which either append a 1 to the composition or increase the last element by 2.

```
App1[l_List]:=Append[l,1];
Inc[l_List]:=Append[Most[l], Last[l]+2];
```

Now we can use the function **Map** to apply these functions to the list of compositions of $n-1$ and $n-2$, respectively.

```
OddComps[n_]:=OddComps[n]=
 Union[Map[App1, OddComps[n-1]], Map[Inc,OddComps[n-2]]]
```

Exercise 3.12 A composition $\sigma = \sigma_1 \cdots \sigma_m$ of n with m parts is said to be *limited* if $1 \le \sigma_i \le n_i$ for all $i = 1, 2, \ldots, n$.

(1) Derive a formula for the generating function for the number of limited compositions of n.

(2) Using Part (1), obtain a simple formula for the case $n_i = k$ for all i.

(3) Prove that the number of limited compositions of n is given by F_{n+1} when $n_i = 2$ for all i.

Solution 3.12 (1) Let $A_m(x)$ be the generating function for the number of limited compositions of n with m parts. Then it is not hard to see that $A_m(x) = \prod_{j=1}^m (x + x^2 + \cdots + x^{n_j})$. Hence, the generating function for the number of limited compositions of n is given by

$$A_{n_1,n_2,\ldots}(x) = 1 + \sum_{m\ge1} A_m(x) = 1 + \sum_{m\ge1}\prod_{j=1}^{m}(x + x^2 + \cdots + x^{n_j}). \qquad (*)$$

(2) Substituting $n_i = k$ in (*) we obtain that

$$A_{k,k,\ldots}(x) = 1 + \sum_{m\ge1}\left(\frac{x - x^{k+1}}{1-x}\right)^m = \frac{1}{1 - \frac{x - x^{k+1}}{1-x}}.$$

This gives the result of Theorem 3.13 for the set $A = [k]$.
(3) Setting $k = 2$ in (*) we get that $A_{2,2,\ldots}(x) = 1/(1 - x - x^2)$. This is the generating function for the sequence F_{n+1} (see A.1) and we recover the result of Theorem 3.10.

Exercise 3.13 Prove that the generating function for the number of compositions of n with exactly k odd parts is given by

$$\frac{x^k(1-x^2)}{(1-2x^2)^{k+1}}.$$

Solution 3.13 Let $A(x,q)$ be the generating function for the number of compositions of n with k odd parts. Our goal is to determine $[q^k]A(x,q)$, the generating function for the number of compositions of n with a fixed number k of odd parts. Since each composition is either empty or starts with an even part or starts with an odd part we obtain that (after appropriate simplification)

$$A(x,q) = 1 + \left(\frac{x^2}{1-x^2} + \frac{qx}{1-x^2}\right) A(x,q)$$