

Chapter Two

Problem 2.1 Under the conditions given in the problem, the Navier-Stokes equations reduce to:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dw}{dr} \right) = \frac{1}{\mu} \frac{dP}{dz} \quad \text{--- (1)}$$

where $\frac{dP}{dz}$ is a constant, since the R.H.S. is a function of z only and the L.H.S. is a function of r only.

The boundary conditions are:

$$\begin{cases} w=0 & \text{at } r=r_0 \end{cases} \quad \text{--- (2)}$$

$$\begin{cases} \frac{dw}{dr}=0 & \text{at } r=0 \end{cases} \quad \text{--- (3)}$$

Integrating (1) twice:

$$w = \frac{1}{4\mu} \frac{dP}{dz} r^2 + C_1 \ln r + C_2 \quad \text{--- (4)}$$

From B.C.s (2) and (3),

$$C_1 = 0, \quad C_2 = -\frac{1}{4\mu} \frac{dP}{dz} r_0^2$$

$$\text{so, } w = \frac{r_0^2}{4\mu} \frac{dP}{dz} \left(\left(\frac{r}{r_0} \right)^2 - 1 \right) \quad \text{--- (5)}$$

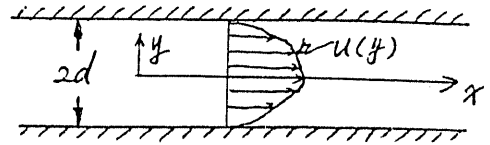
The average velocity:

$$W_m = \frac{1}{\pi r_0^2} \int_0^{r_0} 2\pi r w dr = \frac{r_0^2}{8\mu} \frac{dP}{dz} \quad \text{--- (6)}$$

$$\text{so, } \frac{w}{W_m} = 2 \left(1 - \left(\frac{r}{r_0} \right)^2 \right) \quad \text{--- (7)}$$

Problem 2.2 From the statements of the problem,

$$V=0, W=0, \frac{\partial u}{\partial x}=0, \frac{\partial u}{\partial z}=0.$$



The N-S equations in y- and

z-direction reduce to: $\frac{\partial P}{\partial y}=0$ and $\frac{\partial P}{\partial z}=0$ respectively.

The one in the x-direction reduces to,

$$\frac{dP}{dx} = \mu \frac{d^2 u}{dy^2} \quad \text{--- (1)}$$

We can deduce that $\frac{dP}{dx} = \text{const.}$, since the R.H.S is a function of y only and the L.H.S. is a function of x only.

The boundary conditions are:

$$\begin{cases} u=0 & \text{at } y=d \end{cases} \quad \text{--- (2)}$$

$$\begin{cases} \frac{du}{dy}=0 & \text{at } y=0 \end{cases} \quad \text{--- (3)}$$

Integrating (1) twice,

$$u = \frac{1}{2\mu} \frac{dP}{dx} y^2 + Ay + B \quad \text{--- (4)}$$

From B.C.s (2) and (3),

$$A=0, \text{ and } B = -\frac{1}{2\mu} \frac{dP}{dx} d^2$$

$$\text{So, } u = \frac{1}{2\mu} \frac{dP}{dx} (y^2 - d^2) \quad \text{--- (5)}$$

Problem 2.2 (Continued)

At $y=0$, $u=u_{\max}$,

$$u_{\max} = -\frac{1}{2\mu} \frac{dP}{dx} d^2, \quad \text{--- (6)}$$

so, the velocity distribution in terms of u_{\max} is,

$$\frac{u}{u_{\max}} = 1 - \left(\frac{y}{d}\right)^2 \quad \text{--- (7)}$$

The average velocity is,

$$u_m = \frac{1}{d} \int_0^d u \, dy = -\frac{1}{3\mu} \frac{dP}{dx} d^2 \quad \text{--- (8)}$$

so, the velocity distribution in terms of u_m is,

$$\frac{u}{u_m} = \frac{3}{2} \left[1 - \left(\frac{y}{d}\right)^2 \right] \quad \text{--- (9)}$$

Also,

$$u_{\max} = \frac{3}{2} u_m \quad \text{--- (10)}$$

Problem 2.3 Assume the downward direction to be the positive y -direction. The N-S equation is,

$$\rho \frac{DV}{Dt} = \rho g - \frac{\partial P}{\partial y} + \frac{\partial}{\partial x} \left\{ \mu \left(\frac{\partial V}{\partial x} + \frac{\partial u}{\partial y} \right) \right\} + \frac{\partial}{\partial y} \left\{ \mu \left(2 \frac{\partial V}{\partial y} - \frac{2}{3} \nabla \cdot \vec{V} \right) \right\} + \frac{\partial}{\partial z} \left\{ \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \right\} -$$

Introducing the dimensionless variables:

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{z} = \frac{z}{L}, \quad \bar{u} = \frac{u}{U_0}, \quad \bar{w} = \frac{w}{U_0}$$

$$\bar{P} = \frac{P - P_0}{\rho U_0^2}, \quad \bar{t} = \frac{t U_0}{L}$$

where L is the characteristic length of the problem, U_0 is a reference velocity, and P_0 is the reference pressure.

Sub. all the dimensionless variables into (1), we get,

$$\begin{aligned} \frac{D\bar{V}}{D\bar{t}} = \bar{g} - \frac{\partial \bar{P}}{\partial \bar{y}} + \frac{\partial}{\partial \bar{x}} \left\{ \frac{1}{Re} \left(\frac{\partial \bar{V}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}} \right) \right\} + \frac{\partial}{\partial \bar{y}} \left\{ \frac{1}{Re} \left(2 \frac{\partial \bar{V}}{\partial \bar{y}} - \frac{2}{3} \nabla \cdot \vec{\bar{V}} \right) \right\} \\ + \frac{\partial}{\partial \bar{z}} \left\{ \frac{1}{Re} \left(\frac{\partial \bar{w}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{z}} \right) \right\} \end{aligned}$$

where $\bar{g} = \frac{U_0^2}{gL} = Fr$, called Froude Number,

and $Re = \frac{\rho U_0 L}{\mu}$, called Reynolds Number.

So, the conditions for dynamical similarity are

$$Fr_1 = Fr_2 \quad \text{and} \quad Re_1 = Re_2$$

Problem 2.4 The control volume is as shown in the Fig.

The mass flow rate into the C.V.

through surface (1) is $(\rho V_r) \cdot r \Delta \theta \cdot \Delta z$,

Taking linear approximation, the mass out-flow through surface (2)

is, $(\rho V_r) r \Delta \theta \Delta z + \frac{\partial}{\partial r}(\rho V_r r \Delta \theta \Delta z) \Delta r$.

So, the net mass in-flow is, $-\frac{\partial}{\partial r}(\rho r V_r) \Delta r \Delta \theta \Delta z$

The mass in-flow through surface (3) is $\rho V_z \cdot \Delta r \cdot r \Delta \theta$, and the mass out-flow through surface (4) is,

$\rho V_z r \Delta r \Delta \theta + \frac{\partial}{\partial z}(\rho V_z r \Delta r \Delta \theta) \Delta z$, so the net mass in-flow is $-\frac{\partial}{\partial z}(\rho V_z) r \Delta r \Delta \theta \Delta z$.

Similarly, the net mass in-flow rate in the tangential direction is

$$-\rho V_\theta \Delta r \Delta z - \frac{\partial}{\partial \theta}(\rho V_\theta \Delta r \Delta z) \Delta \theta + \rho V_\theta \Delta r \Delta z = -\frac{\partial}{\partial \theta}(\rho V_\theta) \Delta r \Delta \theta \Delta z$$

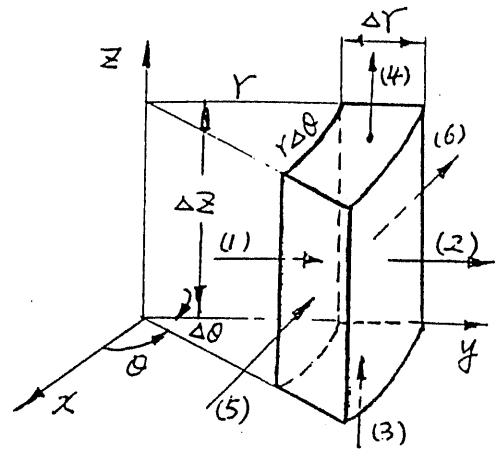
The increase rate of mass in the C.V. is

$$\frac{\partial}{\partial t}(\rho \Delta r \Delta z r \Delta \theta)$$

From the law of mass conservation:

$$\frac{\partial \rho}{\partial t} \Delta r \Delta z \Delta \theta = -\frac{\partial}{\partial r}(\rho r V_r) \Delta r \Delta \theta \Delta z - \frac{\partial}{\partial z}(\rho V_z) r \Delta r \Delta \theta \Delta z - \frac{\partial}{\partial \theta}(\rho V_\theta) \Delta r \Delta \theta \Delta z$$

$$\text{or. } \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(\rho r V_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$$



Problem 2.5 The N-S equation for a fluid of constant viscosity in the x -direction is:

$$\rho \frac{Du}{Dt} = \rho f_x - \frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad \text{--- (1)}$$

For parallel flow, $v \equiv w \equiv 0$, from continuity equation we have:

$$\frac{\partial u}{\partial x} \equiv 0, \quad \text{so, } u = u(y, z, t)$$

From the N-S equations in the y - and z -directions we have $\frac{\partial P}{\partial y} = 0$, and $\frac{\partial P}{\partial z} = 0$, so $P = P(x)$

Therefore, (1) becomes,

$$\rho \frac{\partial u}{\partial t} = \rho f_x - \frac{dP}{dx} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad \text{--- (2)}$$

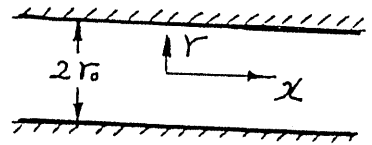
Note that, $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\partial u}{\partial t}$ under the above conditions.

For steady incompressible flow, we have

$$\frac{dP}{dx} = \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho f_x \quad \text{--- (3)}$$

Problem 2.6 (a) From Problem 2.1, for fully developed laminar flow, we have

$$u(r)/u_m = 2 \left(1 - \frac{r^2}{r_0^2}\right)$$



where r_0 is the radius of the pipe, r is the distance from the centre of the pipe and u_m is the mean velocity.

$$d(KE) = d\dot{m} \frac{V^2}{2} = d\dot{m} \left(\frac{u^2 + v^2 + w^2}{2} \right)$$

since $v=0$, $w=0$, $d\dot{m} = \rho 2\pi r u(r) dr$

so $d(KE) = d\dot{m} \frac{V^2}{2} = \rho 2\pi r u(r) \frac{u^2(r)}{2} dr$

$$\begin{aligned} KE &= \int_0^{r_0} \rho \pi r u^3(r) dr \\ &= 8\rho\pi \int_0^{r_0} u_m^3 \left(1 - \frac{r^2}{r_0^2}\right)^3 dr \\ &= -4\pi u_m^3 \int_0^{r_0} \frac{(r_0^2 - r^2)^3}{r_0^6} d(r_0^2 - r^2) \\ &= \rho\pi r_0^2 u_m^3 \end{aligned}$$

The mean kinetic energy is,

$$\overline{KE} = \frac{KE}{\pi r_0^2} = \rho u_m^3$$

(b) For flow between two parallel plates,

$$u = u_m \frac{3}{2} \left(1 - \left(\frac{y}{d}\right)^2\right)$$

$$\begin{aligned} KE &= \int_0^d \rho \cdot 2u \cdot \frac{u^2}{2} dy \\ &= \rho \frac{27}{8} \cdot u_m^3 \int_0^d \left(1 - \frac{y^2}{d^2}\right)^3 dy = \frac{27}{8} \rho d u_m^3 \int_0^1 (1 - \eta^2) d\eta \\ &= 1.543 \rho d u_m^3 \end{aligned}$$

Problem 2.6 (continued)

The mean kinetic energy is,

$$\overline{KE} = KE/2d = 0.7715 \rho U_m^3$$

(c) For turbulent flow through a circular duct

$$U(r) = U_c \left(1 - \frac{r}{r_0}\right)^{1/7} \quad \underline{y = r_0 - r}$$

$$U_m = \frac{1}{\pi r_0^2} \int_0^{r_0} 2\pi r U_c \left(1 - \frac{r}{r_0}\right)^{1/7} dr$$

$$= \frac{2U_c}{r_0^2} \int_0^{r_0} r \left(1 - \frac{r}{r_0}\right)^{1/7} dr$$

$$= \frac{98}{120} U_c$$

$$= 0.8167 U_c$$

$$\underline{U_c = 1.224 U_m}$$

$$\text{So, } U(r) = 1.224 U_m \left(1 - \frac{r}{r_0}\right)^{1/7}$$

$$KE = \int_0^{r_0} \rho \pi r U^3(r) dr$$

$$= 1.8338 \pi U_m^3 \int_0^{r_0} \rho r \left(1 - \frac{r}{r_0}\right)^{3/7} dr$$

$$= \frac{1.8338 \times 49}{170} \rho \pi U_m^3 r_0^2$$

$$= 0.5292 \rho \pi U_m^3 r_0^2$$

$$\overline{KE} = KE/\pi r_0^2 = 0.5292 \rho U_m^3$$

Problem 2.7 From the statements of the problem, the N-S equation in the x-direction simplifies to:

$$g \sin \theta + \nu \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

The boundary conditions are:

$$\begin{cases} u = 0 & \text{at } y = 0 \\ \frac{\partial u}{\partial y} = 0 & \text{at } y = b \end{cases} \quad \text{--- (2)}$$

$$\text{--- (3)}$$

Integrating (1) twice, we have:

$$u = -\frac{g}{2\nu} \sin \theta y^2 + C_1 y + C_2 \quad \text{--- (4)}$$

Using the B.C.s (2) and (3), we get:

$$C_1 = \frac{g}{\nu} \sin \theta b, \quad C_2 = 0$$

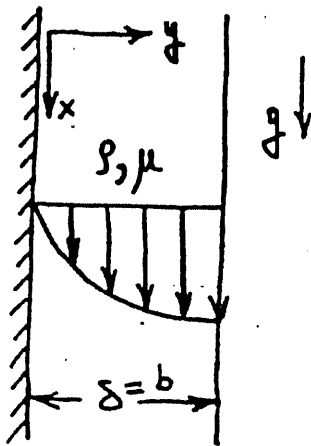
$$\text{so, } u = \frac{g \sin \theta}{2\nu} (2b - y) y \quad \text{--- (5)}$$

$$\begin{aligned} \text{From } Q &= \int_0^b u dy = \frac{g}{2\nu} \sin \theta \int_0^b (2by - y^2) dy \\ &= \frac{g}{3\nu} \sin \theta b^3 \end{aligned} \quad \text{--- (6)}$$

so, the film thickness is:

$$b = \left(\frac{3\nu Q}{g \sin \theta} \right)^{1/3} = \text{const.} \quad \text{--- (7)}$$

Problem 2.7 (for a vertical plate)



Since $\frac{\partial u}{\partial x} = 0$, the continuity eq. gives

$$v = - \int_0^y \frac{\partial u}{\partial x} dy = 0$$

Then, the x-component of the Navier-Stokes equations reduces to

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$p = \text{atm. pressure}$

$$\frac{d^2 u}{dy^2} = - \frac{\rho g}{\mu}$$

$$b = \delta$$

$$\frac{du}{dy} = - \frac{\rho g}{\mu} y + C_1$$

$$u(y) = - \frac{\rho g}{2\mu} y^2 + C_1 y + C_2$$

At $y=0 : u=0 \Rightarrow C_2=0$

At $y=\delta : \tau=0 \Rightarrow \frac{du}{dy}=0 \rightarrow C_1 = \frac{\rho g}{\mu} \delta$

$$u(y) = - \frac{\rho g}{2\mu} y^2 + \frac{\rho g}{\mu} \delta y$$

$$\frac{u(y)}{u_{\max}} = 2 \frac{y}{\delta} - \left(\frac{y}{\delta} \right)^2$$

$$u_{\max} = \frac{\rho g}{2\mu} \delta^2$$

$$Q = \int_0^\delta u dy$$

$$= u_{\max} \delta \int_0^1 (2\eta - \eta^2) d\eta$$

$$= \frac{2}{3} u_{\max} \delta$$

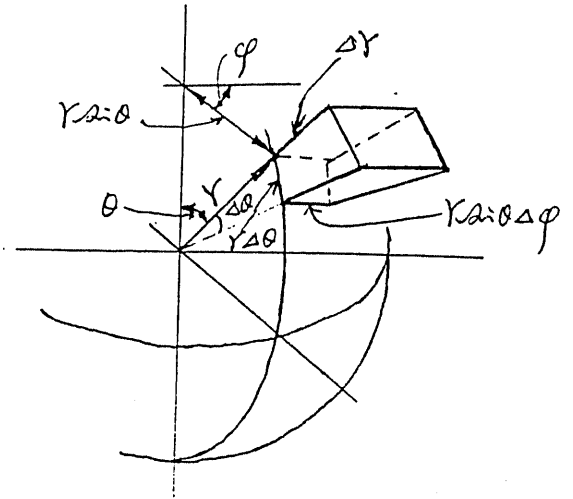
$$= \frac{2}{3} \frac{\rho g}{\mu} \delta^3$$

$$\delta = \left(\frac{3\mu Q}{\rho g} \right)^{1/3}$$

Problem 2.8 As shown in the

Fig., the net mass in-flow rate in the r -direction is:

$$\begin{aligned}\Delta M_r &= \rho V_r \cdot r \Delta \theta \cdot r \sin \theta \Delta \varphi - \left(\rho V_r \cdot r \Delta \theta \cdot r \sin \theta \Delta \varphi + \frac{\partial}{\partial r} (\rho V_r \cdot r \Delta \theta \cdot r \sin \theta \Delta \varphi) \Delta r \right) \\ &= -\frac{\partial}{\partial r} (\rho r^2 V_r) \sin \theta \Delta \theta \Delta \varphi \Delta r\end{aligned}$$



The net mass in-flow rate in θ -direction is:

$$\begin{aligned}\Delta M_\theta &= \rho V_\theta \cdot \Delta r \cdot r \sin \theta \Delta \varphi - \left(\rho V_\theta \Delta r r \sin \theta \Delta \varphi + \frac{\partial}{\partial \theta} (\rho V_\theta \Delta r r \sin \theta \Delta \varphi) \Delta \theta \right) \\ &= -\frac{\partial}{\partial \theta} (\rho V_\theta \sin \theta) \Delta r \Delta \varphi \Delta \theta\end{aligned}$$

The net mass in-flow rate in φ -direction is:

$$\begin{aligned}\Delta M_\varphi &= \rho V_\varphi \cdot \Delta r \cdot r \Delta \theta - \left(\rho V_\varphi \Delta r r \Delta \theta + \frac{\partial}{\partial \varphi} (\rho V_\varphi \Delta r \cdot r \Delta \theta) \Delta \varphi \right) \\ &= -\frac{\partial}{\partial \varphi} (\rho V_\varphi) r \Delta r \Delta \theta \Delta \varphi\end{aligned}$$

The increase rate of mass in the C.V. is:

$$\Delta M_t = \frac{\partial}{\partial t} (\rho \Delta r \cdot r \Delta \theta \cdot r \sin \theta \Delta \varphi) = \frac{\partial}{\partial t} (\rho) r^2 \sin \theta \Delta r \Delta \theta \Delta \varphi$$

From the law of mass conservation: $\Delta M_t = \Delta M_r + \Delta M_\theta + \Delta M_\varphi$

$$\text{So, } \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 V_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho V_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\rho V_\varphi) = 0$$

Problem 2.9 Equation (2.50b) is

$$\rho \frac{Di}{Dt} = \nabla \cdot (k \nabla T) + \frac{Dp}{Dt} + \Phi \quad \text{--- (1)}$$

since

$$di = T ds + \frac{1}{\rho} dp$$

$$ds = \left(\frac{\partial s}{\partial T} \right)_p dT + \left(\frac{\partial s}{\partial p} \right)_T dp$$

From Maxwell's relations:

$$\left(\frac{\partial s}{\partial p} \right)_T = - \left\{ \frac{\partial (1/\rho)}{\partial T} \right\}_p = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial T} \right)_p = - \frac{\beta}{\rho}$$

$$\text{where } \beta = - \frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p$$

For pure substances,

$$\left(\frac{\partial s}{\partial T} \right)_p = \frac{C_p}{T}$$

$$\text{so, } di = C_p dT + \frac{1}{\rho} (1 - \beta T) dp \quad \text{--- (2)}$$

sub (2) into (1), we get,

$$\rho C_p \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + \beta T \frac{Dp}{Dt} + \Phi \quad \text{--- (3)}$$

Problem 2.10 Eq. (2.48) is:

$$\rho \frac{Du}{Dt} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \frac{p}{\rho} \frac{D\rho}{Dt} + \Phi \quad (1)$$

the enthalpy is defined as:

$$i = u + \frac{p}{\rho} \quad (2)$$

$$\text{so, } \frac{Du}{Dt} = \frac{Di}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} + \frac{p}{\rho^2} \frac{D\rho}{Dt} \quad (3)$$

sub. (3) into (1), we get Eq. (2.50a):

$$\rho \frac{Di}{Dt} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \frac{Dp}{Dt} + \Phi$$

Problem 2.11 In problem 2.10, we already reduce the equation to Eq. (2.50a).

(1) since in a solid $u \equiv v \equiv w \equiv 0$, so $\Phi = 0$

(2) In a solid $p = \text{const}$, so $\frac{Dp}{Dt} = 0$

(3) $i = C_p T$, $\frac{Di}{Dt} = \frac{\partial i}{\partial t} = C_p \frac{\partial T}{\partial t}$

so, we have $\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right)$

For $k = \text{const.}$, and steady-state problem, we have the Laplace's equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Problem 2.10 Eq. (2.48) is:

$$\rho \frac{Du}{Dt} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \frac{p}{\rho} \frac{D\rho}{Dt} + \Phi \quad (1)$$

the enthalpy is defined as:

$$i = u + \frac{p}{\rho} \quad (2)$$

$$\text{so, } \frac{Du}{Dt} = \frac{Di}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} + \frac{p}{\rho^2} \frac{D\rho}{Dt} \quad (3)$$

sub. (3) into (1), we get Eq. (2.50a):

$$\rho \frac{Di}{Dt} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \frac{Dp}{Dt} + \Phi$$

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(1) since in a solid $u \equiv v \equiv w \equiv 0$, so $\Phi = 0$

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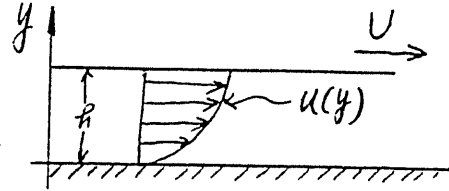
so, we have $\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right)$

For $k = \text{const.}$, and steady-state problem, we have the Laplace's equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Problem 2.12 same as in problem 2.2, the N-S equation in the x-direction can be simplified to:

$$\frac{dP}{dx} = \mu \frac{d^2 u}{dy^2} \quad \text{--- (1)}$$



The B.C.s are :

$$\begin{cases} u=0 & \text{at } y=0 \end{cases} \quad \text{--- (2)}$$

$$\begin{cases} u=U & \text{at } y=h \end{cases} \quad \text{--- (3)}$$

Integrating (1) twice, we get,

$$u = \frac{1}{2\mu} \frac{dP}{dx} y^2 + By + A \quad \text{--- (4)}$$

From (2) & (3), we get,

$$A = 0, \text{ and}$$

$$B = \frac{U}{h} - \frac{h}{2\mu} \frac{dP}{dx} \quad \text{--- (5)}$$

$$\text{so, } u = -\frac{1}{2\mu} \frac{dP}{dx} \frac{y}{h} \left(1 - \frac{y}{h}\right) + \frac{y}{h} U \quad \text{--- (6)}$$

Problem 2.13 When $\frac{dP}{dx} = 0$, we have,

$$\frac{d^2 u}{dy^2} = 0$$

Integrating twice and using the B.C.s, we get,

$$u = \frac{y}{h} U$$

Problem 2.14

$$\begin{aligned}\rho \frac{D}{Dt} \left(\frac{p}{\rho} \right) &= \rho \left(\frac{1}{\rho} \frac{Dp}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt} \right) \\ &= \frac{Dp}{Dt} - \frac{p}{\rho} \frac{D\rho}{Dt}\end{aligned}\quad \text{---(1)}$$

From the continuity equation, we have,

$$\frac{D\rho}{Dt} = -\rho(\text{div } \vec{V}) \quad \text{---(2)}$$

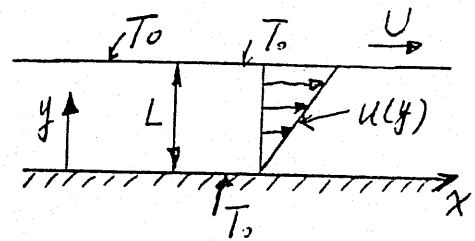
sub (2) into (1), we get:

$$\rho \frac{D}{Dt} \left(\frac{p}{\rho} \right) = \frac{Dp}{Dt} + p(\text{div } \vec{V}) \quad \text{---(3)}$$

i.e.

$$p(\text{div } \vec{V}) = \rho \frac{D}{Dt} \left(\frac{p}{\rho} \right) - \frac{Dp}{Dt} \quad \text{---(4)}$$

Problem 2.15 The velocity components of the flow are,



$$u = U \frac{y}{L}, \quad v = 0$$

We also have $T = T(y)$

so, the energy equation reduces to

$$k \frac{d^2 T}{dy^2} + \Phi = 0 \quad \text{---(1)}$$

Problem 2.15 (cont'd)

where, $\Phi = 2\mu \left\{ \frac{1}{2} \left(\frac{du}{dy} \right)^2 \right\} = \mu \frac{U^2}{L^2}$ — (2)

sub. (2) into (1),

$$\frac{d^2 T}{dy^2} = - \frac{\mu}{k} \frac{U^2}{L^2} \quad \text{--- (3)}$$

The B.C.s are,

$$\begin{cases} T = T_0 & \text{at } y = 0 \end{cases} \quad \text{--- (4)}$$

$$\begin{cases} T = T_0 & \text{at } y = L \end{cases} \quad \text{--- (5)}$$

Integrating (3) twice,

$$T = - \frac{\mu}{2k} \frac{U^2}{L^2} y^2 + C_1 y + C_2 \quad \text{--- (6)}$$

From the B.C.s we obtain,

$$C_2 = T_0, \quad T_0 = \frac{\mu}{2k} \frac{U^2}{L}$$

so, the temperature rise is,

$$\begin{aligned} \Delta T = T - T_0 &= - \frac{\mu}{2k} \frac{U^2}{L^2} y^2 + \frac{\mu}{2k} \frac{U^2}{L} y \\ &= \frac{\mu}{2k} U^2 \left(\frac{y}{L} - \frac{y^2}{L^2} \right) \end{aligned}$$

The maximum temperature rise is,

$$\Delta T|_{y=\frac{L}{2}} = \frac{\mu}{8k} U^2$$